A certain necessary condition of potential blow up for Navier-Stokes equations

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Abstract We show that a necessary condition for T to be a potential blow up time is $\lim\limits_{t\uparrow T}\|v(\cdot,t)\|_{L_3}=\infty$.

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1 Main Result

Consider the Cauchy problem for the classical Navier-Stokes system

$$\partial_t v + v\cdot\nabla v - \Delta v = -\nabla q, \qquad \mathrm{div} v = 0, \tag{1.1}$$

describing the flow of a viscose incompressible fluid in $Q_+=\mathbb{R}^3\times]0,\infty[$ with the initial condition

$$v|_{t=0}=a \tag{1.2}$$

in $\mathbb{R}^3$. Here, as usual, v and q stand for the velocity field and for the pressure field, respectively. For simplicity, let us assume

$$a\in C^\infty_{0,0}(\mathbb{R}^3)\equiv\{v\in C^\infty_0(\mathbb{R}^3):\ \mathrm{div} v=0\}. \tag{1.3}$$

Many years ago, in 1934, J. Leray showed in his celebrated paper [5] that problem (1.1)–(1.3) has at least one weak solution obeying the global energy inequality

$$\frac12\int\limits_{\mathbb{R}^3}|v(x,t)|dx+\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla v|^2dxdt'\leq\frac12\int\limits_{\mathbb{R}^3}|a|^2dx \tag{1.4}$$

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for all positive t. This solution is smooth and unique for sufficiently small values of t. The first instant of time T when singularities occur is called a *blow up* time. By definition, $z_0 = (x_0, t_0)$ is a *regular* point of v if it is essentially bounded in a nonempty parabolic ball with the center at the point z_0 [2] The point z_0 is *singular* if it is not regular.

To the best of our knowledge, it is unknown whether there exists an energy solution to the Cauchy problem (1.1)–(1.3) with a finite time blow up.

However, J. Leray proved certain necessary conditions for T to be a blow up time. They are as follows. Assume that T is a blow up time, then, as it has been shown in the above mentioned paper, for any $3 < m \leq \infty$, there exists a constant c_m, depending on m only, such that

$$\|v(\cdot,t)\|_m \equiv \|v(\cdot,t)\|_{m,\mathbb{R}^3} \equiv \Big(\int\limits_{\mathbb{R}^3} |v(x,t)|^m dx\Big)^{\frac{1}{m}} \geq \frac{c_m}{(T-t)^{\frac{m-3}{2m}}} \tag{1.5}$$

for all $0 < t < T$.

Here, we address the critical case $m = 3$, for which a weaker statement

$$\limsup_{t \to T-0} \|v(\cdot,t)\|_3 = \infty \tag{1.6}$$

has been proven in [1]. The aim of this paper and several previous papers of the author is to improve (1.6). At the moment, the best improvement of (1.6) is given by the following theorem.

Theorem 1.1. *Let v be an energy solution to the Cauchy problem (1.1) and (1.2) with the initial data satisfying (1.3). Let $T > 0$ be a finite blow up time. Then*

$$\lim_{t \to T-0} \|v(\cdot,t)\|_3 = \infty \tag{1.7}$$

holds true.

Now, let us briefly outline a proof that relays upon ideas developed in [8]-[10]. In particular, in [8], a certain type of scaling has been invented, which, after passing to the limit, gives a special non-trivial solution to the Navier-Stokes equations provided there is a finite time blow up. In [9] and [10], it

[2] $Q(z_0, r) = B(x_0, r) \times]t_0 - r^2, t_0[$ is a parabolic ball of radius r centered at the point z_0.

has been shown that the same type of scaling and blowing-up can produce the so-called Lemarie-Rieusset local energy solutions. It turns out to be that the backward uniqueness technique is still applicable to them. Although the theory of backward uniqueness itself is relatively well understood, its realization is not an easy task and based on delicate regularity results for the Navier-Stokes equations. Actually, there are two main points to verify: solutions, produced by scaling and blowing-up, vanish at the last moment of time and have a spatial decay. The first property is easy when working with L_3-norm while the second one is harder. However, under certain restrictions, the required decay is a consequence of the Lemarie-Rieusset theory. So, the main technical part of the whole procedure is to show that scaling and blowing-up lead to local energy solutions. On that way, a lack of compactness of initial data of scaled solutions in $L_{2,\mathrm{loc}}$ is the main obstruction. This is why the same theorem for a stronger scale-invariant norm of the space $H^{\frac{1}{2}}$ is easier. The reason for that is a compactness of the corresponding embedding, see [7] and [9].

In this paper, we are going to show that, despite of a lack of compactness in L_3-case, the limit of the sequence of scaled solutions is still a local energy solution, for which a spatial decay takes place. Technically, this can be done by splitting each scaled solution into two parts. The first one is a solution to a non-linear problem but with zero initial data while the second one is a solution of a linear problem with weakly converging nonhomogeneous initial data.

2 Estimates of Scaled Solutions

Assume that our statement is false and there exists an increasing sequence t_k converging to T as $k \to \infty$ such that

$$\sup_{k\in\mathbb{N}} \|v(\cdot,t_k)\|_3 = M < \infty. \tag{2.1}$$

By the definition of a blow up time for energy solutions, there exists at least one singular point at time T. Without loss of generality, we may assume that it is $(0,T)$. Moreover, the blow-up profile has the finite L_3-norm, i.e.,

$$\|v(\cdot,T)\|_3 < \infty. \tag{2.2}$$

Let us scale v and q so that

$$u^{(k)}(y,s)=\lambda_k v(x,t),\quad p^{(k)}(y,s)=\lambda_k^2 q(x,t),\quad (y,s)\in\mathbb{R}^3\times]-\lambda_k^{-2}T,0[,\tag{2.3}$$

where

$$x=\lambda_k y,\qquad t=T+\lambda_k^2 s,$$

$$\lambda_k=\sqrt{\frac{T-t_k}{S}}$$

and a positive parameter $S<10$ will be defined later.

By the scale invariance of L_3-norm, $u^{(k)}(\cdot,-S)$ is uniformly bounded in $L_3(\mathbb{R}^3)$, i.e.,

$$\sup_{k\in\mathbb{N}}\|u^{(k)}(\cdot,-S)\|_3=M<\infty.\tag{2.4}$$

Let us decompose our scaled solution $u^{(k)}$ into two parts: $u^{(k)}=v^{(k)}+w^{(k)}$. Here, $w^{(k)}$ is a solution to the Cauchy problem for the Stokes system:

$$\partial_t w^{(k)}-\Delta w^{(k)}=-\nabla r^{(k)},\quad \operatorname{div} w^{(k)}=0\quad\text{in}\quad \mathbb{R}^3\times]-S,0[,$$
$$w^{(k)}(\cdot,-S)=u^{(k)}(\cdot,-S).\tag{2.5}$$

Apparently, (2.5) can be reduced to the Cauchy problem for the heat equation so that the pressure $r^{(k)}=0$ and $w^{(k)}$ can be worked out with the help of the heat potential. The estimate below is well-known, see, for example [2],

$$\sup_k\{\|w^{(k)}\|_{L_5(\mathbb{R}^3\times]-S,0[}+\|w^{(k)}\|_{L_{3,\infty}(\mathbb{R}^3\times]-S,0[}\}\leq c(M)<\infty.\tag{2.6}$$

It is worthy to note that, by the scale invariance, $c(M)$ in (2.6) is independent of S.

As to $v^{(k)}$, it is a solution to the Cauchy problem for the following perturbed Navier-Stokes system

$$\partial_t v^{(k)}+\operatorname{div}(v^{(k)}+w^{(k)})\otimes(v^{(k)}+w^{(k)})-\Delta v^{(k)}=-\nabla p^{(k)},$$
$$\operatorname{div} v^{(k)}=0\quad\text{in}\quad\mathbb{R}^3\times]-S,0[,\tag{2.7}$$
$$v^{(k)}(\cdot,-S)=0.$$

Now, our aim is to show that, for a suitable choice of $-S$, we can prove unform estimates of $v^{(k)}$ and $p^{(k)}$ in certain spaces, pass to the limit as $k\to\infty$, and conclude that the limit functions u and p are a local energy solution to

the Cauchy problem for the Navier-Stokes system in $\mathbb{R}^3\times]-S,0[$ associated the initial data, generated by the weak L_3-limit of the sequence $u^{(k)}(\cdot,-S)$.

Let us start with estimates of solution to (2.7). First of all, we know the formula for the pressure:

$$p^{(k)}(x,t)=-\frac{1}{3}|u^{(k)}(x,t)|^2+\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}K(x-y):u^{(k)}(y,t)\otimes u^{(k)}(y,t)dy, \quad (2.8)$$

where $K(x)=\nabla^2(1/|x|)$.

Next, we may decompose the pressure in the same way as it has been done in [6]. For $x_0\in\mathbb{R}^3$ and for $x\in B(x_0,3/2)$, we let

$$p^{(k)}_{x_0}(x,t)\equiv p^{(k)}(x,t)-c^{(k)}_{x_0}(t)=p^{1(k)}_{x_0}(x,t)+p^{2(k)}_{x_0}(x,t), \quad (2.9)$$

where

$$p^{1(k)}_{x_0}(x,t)=-\frac{1}{3}|u^{(k)}(x,t)|^2+\frac{1}{4\pi}\int\limits_{B(x_0,2)}K(x-y):u^{(k)}(y,t)\otimes u^{(k)}(y,t)dy,$$

$$p^{2(k)}_{x_0}(x,t)=\frac{1}{4\pi}\int\limits_{\mathbb{R}^3\setminus B(x_0,2)}(K(x-y)-K(x_0-y)):u^{(k)}(y,t)\otimes u^{(k)}(y,t)dy,$$

$$c^{(k)}_{x_0}(t)=\frac{1}{4\pi}\int\limits_{\mathbb{R}^3\setminus B(x_0,2)}K(x_0-y):u^{(k)}(y,t)\otimes u^{(k)}(y,t)dy.$$

Using the similar arguments as in [4], one can derive estimates for the above counterparts of the pressure. Here, they are:

$$\|p^{1(k)}_{x_0}(\cdot,t)\|_{L_{\frac{3}{2}}(B(x_0,3/2))}\leq c(M)(\|v^{(k)}(\cdot,t)\|^2_{L_3(B(x_0,2))}+1), \quad (2.10)$$

$$\sup_{B(x_0,3/2)}|p^{2(k)}_{x_0}(x,t)|\leq c(M)(\|v^{(k)}(\cdot,t)\|^2_{L_{2,\mathrm{unif}}}+1), \quad (2.11)$$

where

$$\|g\|_{L_{2,\mathrm{unif}}}=\sup_{x_0\in\mathbb{R}^3}\|g\|_{L_2(B(x_0,1))}.$$

We further let

$$\alpha(s)=\alpha(s;k,S)=\|v^{(k)}(\cdot,s)\|^2_{2,\mathrm{unif}},$$

$$\beta(s)=\beta(s;k,S)=\sup_{x\in R^3}\int\limits_{-S}^{s}\int\limits_{B(x,1)}|\nabla v^{(k)}|^2dyd\tau.$$

From (2.10), (2.11), we find the estimate of the scaled pressure

$$\delta(0)\leq c(M)\Big[\gamma(0)+\int\limits_{-S}^{0}(1+\alpha^{\frac{3}{2}}(s))ds\Big], \tag{2.12}$$

with some positive constant $c(M)$ independent of k and S. Here, γ and δ are defined as

$$\gamma(s)=\gamma(s;k,S)=\sup_{x\in R^3}\int\limits_{-S}^{s}\int\limits_{B(x,1)}|v^{(k)}(y,\tau)|^3dyd\tau$$

and

$$\delta(s)=\delta(s;k,S)=\sup_{x\in R^3}\int\limits_{-S}^{s}\int\limits_{B(x,3/2)}|p^{(k)}(y,\tau)-c^{(k)}_x(\tau)|^{\frac{3}{2}}dy\,d\tau,$$

respectively. It is known that an upper bound for γ can be given by the known multiplicative inequality

$$\gamma(s)\leq c\Big(\int\limits_{-S}^{s}\alpha^3(\tau)d\tau\Big)^{\frac{1}{4}}\Big(\beta(s)+\int\limits_{-S}^{s}\alpha(\tau)d\tau\Big)^{\frac{3}{4}}. \tag{2.13}$$

Fix $x_0\in\mathbb{R}^3$ and a smooth non-negative function φ such that

$$\varphi=1\quad\text{in}\quad B(1),\qquad \text{spt}\,\subset B(3/2)$$

and let $\varphi_{x_0}(x)=\varphi(x-x_0)$.

Since the function $v^{(k)}$ is smooth on $[-S,0[$, all our further actions are going to be legal. In particular, we may write down the following energy identity

$$\int\limits_{\mathbb{R}^3}\varphi^2_{x_0}(x)|v^{(k)}(x,s)|^2dx+2\int\limits_{-S}^{s}\int\limits_{\mathbb{R}^3}\varphi^2_{x_0}|\nabla v^{(k)}|^2dxd\tau=$$

$$= \int\limits_{-S}^{s}\int\limits_{\mathbb{R}^3}\Big[|v^{(k)}|^2\Delta\varphi^2_{x_0}+v^{(k)}\cdot\nabla\varphi^2_{x_0}(|v^{(k)}|^2+2p^{(k)}_{x_0})\Big]dxd\tau+$$

$$+\int\limits_{-S}^{s}\int\limits_{\mathbb{R}^3}\Big[w^{(k)}\cdot\nabla\varphi^2_{x_0}|v^{(k)}|^2+2\varphi^2_{x_0}w^{(k)}\otimes(w^{(k)}+v^{(k)}):\nabla v^{(k)}+$$

$$+2w^{(k)}\cdot v^{(k)}(w^{(k)}+v^{(k)})\cdot\nabla\varphi^2_{x_0}\Big]dxd\tau=I_1+I_2.$$

The first term I_1 is estimated with the help of the Hölder inequality, multiplicative inequality (2.13), and bounds (2.10), (2.11). So, we find

$$I_1\leq c(M)\Big[\int\limits_{-S}^{s}(1+\alpha(\tau)+\alpha^{\frac{3}{2}}(\tau))d\tau+$$

$$+\Big(\int\limits_{-S}^{s}\alpha^3(\tau)d\tau\Big)^{\frac{1}{4}}\Big(\beta(s)+\int\limits_{-S}^{s}\alpha(\tau)d\tau\Big)^{\frac{3}{4}}\Big].$$

Now, let us evaluate the second term

$$I_2\leq c\int\limits_{-S}^{s}\|v^{(k)}(\cdot,\tau)\|^2_{L_3(B(x_0,3/2))}\|w^{(k)}(\cdot,\tau)\|_{L_3(B(x_0,3/2))}d\tau+$$

$$+c\int\limits_{-S}^{s}\Big(\int\limits_{B(x_0,3/2)}|w^{(k)}|^5dx\Big)^{\frac{1}{5}}\Big(\int\limits_{B(x_0,3/2)}|v^{(k)}|^{\frac{5}{4}}|\nabla v^{(k)}|^{\frac{5}{4}}dx\Big)^{\frac{4}{5}}d\tau+$$

$$+c\beta^{\frac{1}{2}}(s)\Big(\int\limits_{-S}^{s}\int\limits_{B(x_0,3/2)}|w^{(k)}|^4dxd\tau\Big)^{\frac{1}{2}}d\tau+$$

$$+c\int\limits_{-S}^{s}\|v^{(k)}(\cdot,\tau)\|_{L_3(B(x_0,3/2))}\|w^{(k)}(\cdot,\tau)\|^2_{L_3(B(x_0,3/2))}d\tau.$$

Taking into account and applying Hölder inequality several times(2.6), we find

$$I_2\leq c(M)\gamma^{\frac{2}{3}}(s)(s+S)^{\frac{1}{3}}+$$

$$+c\int\limits_{-S}^{s}\Big(\int\limits_{B(x_0,3/2)}|w^{(k)}|^5dx\Big)^{\frac15}\Big(\int\limits_{B(x_0,3/2)}|\nabla v^{(k)}|^2dx\Big)^{\frac12}\times$$

$$\times\Big(\int\limits_{B(x_0,3/2)}|v^{(k)}|^{\frac{10}{3}}dx\Big)^{\frac{3}{10}}d\tau+c(M)\beta^{\frac12}(s)(s+S)^{\frac{1}{10}}+$$

$$+c(M)\gamma^{\frac13}(s)(s+S)^{\frac23}.$$

It remains to use another known multiplicative inequality

$$\Big(\int\limits_{B(x_0,3/2)}|v^{(k)}(x,s)|^{\frac{10}{3}}dx\Big)^{\frac{3}{10}}\leq c\Big(\int\limits_{B(x_0,3/2)}|v^{(k)}(x,s)|^2dx\Big)^{\frac15}\times$$

$$\times\Big(\int\limits_{B(x_0,3/2)}(|\nabla v^{(k)}(x,s)|^2+|v^{(k)}(x,s)|^2dx\Big)^{\frac{3}{10}}$$

and to conclude that

$$I_2\leq c(M)\gamma^{\frac23}(s)(s+S)^{\frac13}+c(M)\beta^{\frac12}(s)(s+S)^{\frac{1}{10}}+c(M)\gamma^{\frac13}(s)(s+S)^{\frac23}+$$

$$+c\Big(\beta(s)+\int\limits_{-S}^{s}\alpha(\tau)d\tau\Big)^{\frac45}\times\Big(\int\limits_{-S}^{s}\alpha(\tau)\|w^{(k)}(\cdot,\tau)\|^5_{L_{5,\mathrm{unif}}}d\tau\Big)^{\frac15}.$$

Finally, we find

$$\alpha(s)+\beta(s)\leq c(M)\Big[(s+S)^{\frac15}+$$

$$+\int\limits_{-S}^{s}\Big(\alpha(\tau)(1+\|w^{(k)}(\cdot,\tau)\|^5_{L_{5,\mathrm{unif}}})+\alpha^3(\tau)\Big)\,d\tau\Big],\tag{2.14}$$

which is valid for any $s\in[-S,0[$ and for some positive constant $c(M)$ independent of k, s, and S.

It is not so difficult to show that there is a positive constant $S(M)$ such that

$$\alpha(s)\leq\frac{1}{10}\tag{2.15}$$

for any $s\in]-S(M),0[$. In turn, the latter will also imply that

$$\alpha(s)\leq c(M)(s+S)^{\frac15}\tag{2.16}$$

for any $s \in]-S(M),0[$.

To see how this can be worked out, let us assume

$$\alpha(s) \leq 1 \tag{2.17}$$

for $-S \leq s < s_0 \leq 0$. Then (2.14) yields

$$\alpha(s) \leq c(M)((s+S)^{\frac{1}{5}} + y(s)) \tag{2.18}$$

for the same s. Here,

$$y(s) = \int\limits_{-S}^{s} \alpha(\tau)(2+g(\tau))d\tau, \qquad g(s) = \|w^{(k)}(\cdot,s)\|^2_{L_5(\mathbb{R}^3)}.$$

The function $y(s)$ obeys the differential inequality

$$y'(s) \leq c(M)(2+g(s))((s+S)^{\frac{1}{5}} + y(s)) \tag{2.19}$$

for $-S \leq s < s_0 \leq 0$. After integrating (2.19), we find

$$y(s) \leq c(M)\int\limits_{-S}^{s} \Big((\tau+S)^{\frac{1}{5}}(2+g(\tau))\exp\Big\{c(M)\int\limits_{\tau}^{s}(2+g(\vartheta))\Big\}d\vartheta\Big)d\tau \tag{2.20}$$

for $-S \leq s < s_0 \leq 0$. Taking into account estimate (2.6), we derive from (2.20) the following bound

$$y(s) \leq c_1(M)(s+S)^{\frac{1}{5}} \tag{2.21}$$

for $-S \leq s < s_0 \leq 0$ and thus

$$y(s) \leq c_1(M)S^{\frac{1}{5}} \tag{2.22}$$

for the same s.

Now, let us pick up $S(M) > 0$ so small that

$$c(M)(1+c_1(M))S^{\frac{1}{5}}(M) = \frac{1}{20}. \tag{2.23}$$

We claim that, for such a choice of S(M), statement (2.15) holds true. Indeed, assume that it is false. Then since $\alpha(s)$ is a continuous function on $[-S,0[$

and $\alpha(0) = 0$, there exists $s_0 \in]-S, 0[$ such that $0 \leq \alpha(s) < \frac{1}{10}$ for all $s \in]-S, s_0[$ and $\alpha(s_0) = \frac{1}{10}$. In this case, we may use first (2.22) and then (2.18), (2.23) to get

$$\alpha(s) \leq c(M)(1 + c_1(M))S^{\frac{1}{5}}(M) = \frac{1}{20}$$

for $s \in]-S, s_0[$. This leads to a contradiction and, hence, (2.15) has been proven. It remains to use (2.18) and (2.21) with $s_0 = 0$ in order to establish (2.16).

3 Limiting Procedure

As to $w^{(k)}$, it is defined by the solution formula

$$w^{(k)}(x,t) = \frac{1}{(4\pi(s+S))^{\frac{3}{2}}} \int\limits_{\mathbb{R}^3} \exp\Big(- \frac{|x-y|^2}{4(s+S)}\Big) u^{(k)}(y,-S)dy.$$

Moreover, by standard localization arguments, the following estimate can be derived:

$$\sup_{-S<s<0} \sup_{x_0 \in \mathbb{R}^3} \|w^{(k)}(\cdot,s)\|^2_{L_2(B(x_0,1))} +$$

$$+ \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-S}^{0} \int\limits_{B(x_0,1)} |\nabla w^{(k)}(y,s)|^2 dyds \leq c(M) < \infty.$$

Obviously, $w^{(k)}$ and all its derivatives converge to w and to its corresponding derivatives uniformly in sets of the form $\overline{B}(R) \times [\delta, 0]$ for any $R > 0$ and for any $\delta \in]-S, 0[$. The limit function satisfies the same representation formula

$$w(x,t) = \frac{1}{(4\pi(s+S))^{\frac{3}{2}}} \int\limits_{\mathbb{R}^3} \exp\Big(- \frac{|x-y|^2}{4(s+S)}\Big) a_0(y)dy,$$

in which a_0 is the weak $L_3(\mathbb{R}^3)$-limit of the sequence $u^{(k)}(\cdot,-S)$. The function w satisfies the uniform local energy estimate

$$\sup_{-S<s<0} \sup_{x_0 \in \mathbb{R}^3} \|w(\cdot,s)\|^2_{L_2(B(x_0,1))} +$$

$$+\sup_{x_0\in\mathbb{R}^3}\int\limits_{-S}^{0}\int\limits_{B(x_0,1)}|\nabla w(y,s)|^2dyds\leq c(M)<\infty.$$

The important fact, coming from the solution formula, is as follows:

$$w\in C([-S,0];L_3(\mathbb{R}^3))\cap L_5(\mathbb{R}^3\times]-S,0[). \tag{3.1}$$

Next, the uniform local energy estimate for the sequence $u^{(k)}$ (with respect to k) can be deduced from the estimates above. This allows us to exploit the limiting procedure explained in [6] in details. As a result, one can selected a subsequence, still denoted by $u^{(k)}$, with the following properties:

for any $a>0$,

$$u^{(k)}\to u \tag{3.2}$$

weakly-star in $L_\infty(-S,0;L_2(B(a)))$ and strongly in $L_3(B(a)\times]-S,0[)$ and in $C([\tau,0];L_{\frac{9}{8}}(B(a)))$ for any $-S<\tau<0$;

$$\nabla u^{(k)}\to\nabla u \tag{3.3}$$

weakly in $L_2(B(a)\times]-S,0[)$;

$$t\mapsto\int\limits_{B(a)}u^{(k)}(x,t)\cdot w(x)dx\to t\mapsto\int\limits_{B(a)}u(x,t)\cdot w(x)dx \tag{3.4}$$

strongly in $C([-S,0])$ for any $w\in L_2(B(a))$. The corresponding sequences $v^{(k)}$ and $w^{(k)}$ converge to their limits v and w in the same sense and of course $u=v+w$. For the pressure p, we have the following convergence: for any $n\in\mathbb{N}$, there exists a sequences $c^{(k)}_n\in L_{\frac{3}{2}}(-S,0)$ such that

$$\widetilde{p}^{(k)}_n\equiv p^{(k)}-c^{(k)}_n\rightharpoonup p \tag{3.5}$$

in $L_{\frac{3}{2}}(-S,0;L_{\frac{3}{2}}(B(n)))$.

So, arguing in the same way as in [6], one can show that u and p satisfy the following conditions:

$$\sup_{-S<s<0}\sup_{x_0\in\mathbb{R}^3}\|u(\cdot,s)\|^2_{L_2(B(x_0,1))}+\sup_{x_0\in\mathbb{R}^3}\int\limits_{-S}^{0}\int\limits_{B(x_0,1)}|\nabla u(y,s)|^2dyds<\infty; \tag{3.6}$$

$$p \in L_{\frac{3}{2}}(-S,0;L_{\frac{3}{2},\mathrm{loc}}(\mathbb{R}^3); \tag{3.7}$$

the function

$$s \mapsto \int\limits_{\mathbb{R}^3} u(y,s)\cdot w(y)dy \tag{3.8}$$

is continuous on $[-S,0]$ for any compactly supported $w \in L_2(\mathbb{R}^3)$;

$$\partial_t u + u\cdot\nabla u - \Delta u = -\nabla p, \quad \operatorname{div} u = 0 \tag{3.9}$$

in $\mathbb{R}^3\times]-S,0[$ in the sense of distributions;

for any $x_0 \in \mathbb{R}^2$, there exists a function $c_{x_0} \in L_{\frac{3}{2}}(-S,0)$ such that

$$p(x,t) - c_{x_0}(t) = p^1_{x_0}(x,t) + p^2_{x_0}(x,t) \tag{3.10}$$

for all $x \in B(x_0,3/2)$;

for any $s \in]-S,0[$ and for $\varphi \in C^\infty_0(\mathbb{R}^3\times]-S,S[)$,

$$\int\limits_{\mathbb{R}^3} \varphi^2(y,s)|u(y,s)|^2dy + 2\int\limits_{-S}^{s}\int\limits_{\mathbb{R}^3}\varphi^2|\nabla u|^2dyd\tau \leq$$

$$\leq \int\limits_{-S}^{s}\int\limits_{\mathbb{R}^3}\Big(|u|^2(\Delta\varphi^2+\partial\varphi^2)+u\cdot\nabla\varphi^2(|u|^2+2p)\Big)dyd\tau. \tag{3.11}$$

Passing to the limit in (2.16), we find

$$\sup_{x_0\in\mathbb{R}^3}\|v(\cdot,s)\|^2_{L_2(B(x_0,1))} \leq c(M)(s+S)^{\frac{1}{5}}$$

for all $s \in [-S,0]$. And thus

$$v \to 0 \qquad \text{in} \quad L_{2,\mathrm{loc}}(\mathbb{R}^3)$$

as $s \downarrow -S$. Then, taking into account (3.1), we can conclude that

$$u \to a_0 \qquad \text{in} \quad L_{2,\mathrm{loc}}(\mathbb{R}^3). \tag{3.12}$$

as $s \downarrow -S$.

By definition accepted in [6], the pair u and p, satisfying (3.6)–(3.12), is a local energy solution to the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3\times]-S,0[$ associated with the initial velocity a_0.

Now, our aim is to show that u is not identically zero. Using the inverse scaling, we observe that the following identity takes place:

$$\frac{1}{a^2}\int\limits_{Q(a)}(|u^{(k)}|^3+|\widetilde{p}^{(k)}|^{\frac32})dy\,ds=\frac{1}{(a\lambda_k)^2}\int\limits_{Q(z_T,a\lambda_k)}(|v|^3+|q-b^{(k)}|^{\frac32})dx\,dt$$

for all $0<a<a_*=\inf\{1,\sqrt{S/10},\sqrt{T/10}\}$ and for all $\lambda_k\leq 1$. Here, $z_T=(0,T)$, $\widetilde{p}^{(k)}\equiv\widetilde{p}_2^{(k)}$, and $b^{(k)}(t)=\lambda_k^{-2}c_2^{(k)}(s)$. Since the pair v and $q-b^{(k)}$ is a suitable weak solution to the Navier-Stokes equations in $Q(z_T,\lambda_k a_*)$, we find

$$\frac{1}{a^2}\int\limits_{Q(a)}(|u^{(k)}|^3+|\widetilde{p}^{(k)}|^{\frac32})dy\,ds>\varepsilon \tag{3.13}$$

for all $0<a<a_*$ with a positive universal constant ε.

Now, by (3.2) and (3.5),

$$\frac{1}{a^2}\int\limits_{Q(a)}|u^{(k)}|^3dy\,ds\to\frac{1}{a^2}\int\limits_{Q(a)}|u|^3dy\,ds \tag{3.14}$$

for all $0<a<a_*$ and

$$\sup_{k\in\mathbb{N}}\frac{1}{a_*^2}\int\limits_{Q(a_*)}(|u^{(k)}|^3+|\widetilde{p}^{(k)}|^{\frac32})dy\,ds=M_1<\infty. \tag{3.15}$$

To treat the pressure $\widetilde{p}^{(k)}$, we do the usual decomposition of it into two parts, see similar arguments in [9]. The first one is completely controlled by the pressure while the second one is a harmonic function in $B(a_*)$ for all admissible t. In other words, we have

$$\widetilde{p}^{(k)}=p_1^{(k)}+p_2^{(k)}$$

where $p_1^{(k)}$ obeys the estimate

$$\|p_1^{(k)}(\cdot,s)\|_{\frac32,B(a_*)}\leq c\|u^{(k)}(\cdot,s)\|^2_{3,B(a_*)}. \tag{3.16}$$

For the harmonic counterpart of the pressure, we have

$$\sup_{y\in B(a_*/2)}|p_2^{(k)}(y,s)|^{\frac32}\leq c(a_*)\int\limits_{B(a_*)}|p_2^{(k)}(y,s)|^{\frac32}dy$$

$$\leq c(a_*)\int\limits_{B(a_*)}(|\widetilde{p}^{(k)}(y,s)|^{\frac{3}{2}}+|u^{(k)}(y,s)|^3)dy \tag{3.17}$$

for all $-a_*^2 < s < 0$.

For any $0 < a < a_*/2$,

$$\varepsilon \leq \frac{1}{a^2}\int\limits_{Q(a)}(|\widetilde{p}^{(k)}|^{\frac{3}{2}}+|u^{(k)}|^3)dy\,ds \leq$$

$$\leq c\frac{1}{a^2}\int\limits_{Q(a)}(|p_1^{(k)}|^{\frac{3}{2}}+|p_2^{(k)}|^{\frac{3}{2}}+|u^{(k)}|^3)dy\,ds \leq$$

$$\leq c\frac{1}{a^2}\int\limits_{Q(a)}(|p_1^{(k)}|^{\frac{3}{2}}+|u^{(k)}|^3)dy\,ds+$$

$$+ca^3\frac{1}{a^2}\int\limits_{-a^2}^{0}\sup_{y\in B(a_*/2)}|p_2^{(k)}(y,s)|^{\frac{3}{2}}ds.$$

From (3.15)–(3.17), it follows that

$$\varepsilon \leq c\frac{1}{a^2}\int\limits_{Q(a_*)}|u^{(k)}|^3dy\,ds + ca\int\limits_{-a^2}^{0}ds\int\limits_{B(a_*)}(|\widetilde{p}^{(k)}(y,s)|^{\frac{3}{2}}+|u^{(k)}(y,s)|^3)dy \leq$$

$$\leq c\frac{1}{a^2}\int\limits_{Q(a_*)}|u^{(k)}|^3dy\,ds + ca\int\limits_{Q(a_*)}(|\widetilde{p}^{(k)}|^{\frac{3}{2}}+|u^{(k)}|^3)dy\,ds \leq$$

$$\leq c\frac{1}{a^2}\int\limits_{Q(a_*)}|u^{(k)}|^3dy\,ds + cM_1aa_*^2$$

for all $0 < a < a_*/2$. After passing to the limit and picking up sufficiently small a, we find

$$0 < c\varepsilon a^2 \leq \int\limits_{Q(a_*)}|u|^3dy\,ds \tag{3.18}$$

for some positive $0 < a < a_*/2$. So, the limit function u is non-trivial.

PROOF THEOREM 1.1 Since the limit function $a_0 \in L_3$,

$$\|a_0\|_{2,B(x_0,1)} \to 0$$

as $|x_0| \to \infty$. The latter, together with Theorem 1.4 from [6] and ε-regularity theory for the Navier-Stokes equations, gives required decay at infinity. The last thing to be noticed is that the following important property holds true:

$$u(\cdot, 0) = 0. \tag{3.19}$$

This follows from (2.2) and (3.2), see the last statement in (3.2). More details on the matter can be found in papers [8] and [9]. According to backward uniqueness for the Navier-Stokes, $u(\cdot, s) = 0$ for any $s \in]-a_*^2, 0[$, which contradicts (3.18). So, z_T is not a singular point. Theorem 1.1 is proved.
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